Again, the normal at P to the path of P, $\rho = f(\theta)$, must also pass through C. Hence the angle OPC is the complement of the angle ψ between OP and the tangent at P. Hence,

$$\tan OPC = \cot \psi = \frac{d\rho}{\rho d\theta}.$$

(See any book on Calculus.) Therefore, from the right triangle COP, we have

$$OC = \rho \tan OPC = \frac{d\rho}{d\theta} = \frac{d}{d\theta} f(\theta).$$

Also solved by J. B. REYNOLDS and W. E. CEDERBERG.

257 (Number Theory). Proposed by LOUIS O'SHAUGHNESSY, University of Pennsylvania.

Find a general expression for the number of positive integers from 1 to 10^t, inclusive, every one of which contains the figure 9 exactly r times $(0 \leqq r \leqq t)$.

SOLUTION BY THE PROPOSER.

In the case of the integers from 1 to 10, we have nine which do not contain the figure 9 and one which contains one 9. This shall be indicated by the expression $9 + 1$.

In the case of 10^2, the number of integers, which do not contain 9, is $9 \cdot 9$, or 9^2; which contain one 9, is $9 \cdot 1 + 9$, or $2 \cdot 9$; which contain two 9's, is 1, and we have the expansion of

$$(9 + 1)^2 = 9^2 + 2 \cdot 9 + 1.$$

For 10^3, we have $9 \cdot 9^2$, $9 \cdot 2 \cdot 9 + 9^2$, $9 \cdot 1 + 2 \cdot 9$, and 1, or $9^3 + 3 \cdot 9^2 + 3 \cdot 9 + 1$.

Then, for 10^k, assume the expansion of $(9 + 1)^k$, or

$$9^k + \binom{k}{1} 9^{k-1} + \binom{k}{2} 9^{k-2} + \cdots + \binom{k}{n-1} 9^{k-(n-1)} + \binom{k}{n} 9^{k-n} + \cdots + \binom{k}{k-1} 9 + 1.$$

For 10^{k+1} we reason as follows: The number of integers which do not contain 9 is $9 \cdot 9^k$, or 9^{k+1}; which contain one 9, is $9 \cdot \binom{k}{1} 9^{k-1} + 9^k$, or $\binom{k+1}{1} 9^k$; which contain two 9's, is $9 \cdot \binom{k}{2} 9^{k-2} + \binom{k}{1} 9^{k-1}$ or $\binom{k+1}{2} 9^{k-1}$, and which contain n 9's, is

$$9 \cdot \binom{k}{n} 9^{k-n} + \binom{k}{n-1} 9^{k-(n-1)} = \left[\binom{k}{n} + \binom{k}{n-1}\right] 9^{k-n+1} = \binom{k+1}{n} 9^{k+1-n}.$$

Hence, we have, for 10^{k+1}, the expansion of $(9 + 1)^{k+1}$, or

$$9^{k+1} + \binom{k+1}{1} 9^k + \cdots + \binom{k+1}{n} 9^{k+1-n} + \cdots + \binom{k+1}{k} 9 + 1.$$

Now, the derived expression holds for $k = 2$ and for $k = 3$; hence it holds for all positive integral values of k.

Therefore, the general expression required is $\binom{t}{r} 9^{t-r}$.

Also solved by HORACE OLSON, H. C. FEEMSTER, C. C. YEN, and N. P. PANDYA.

258 (Number Theory). Proposed by A. A. BENNETT, University of Texas.

Find a recursion formula in terms of binomial coefficients for a_n, where the a's are defined by the condition that the persymmetric determinants

$$\begin{vmatrix} a_0 & a_1 & a_2 & \cdot & \cdot \\ a_1 & a_2 & \cdot & \cdot & \cdot \\ a_2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a_{n-1} \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a_1 & a_2 & a_3 & \cdot & \cdot \\ a_2 & a_3 & \cdot & \cdot & \cdot \\ a_3 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a_n \end{vmatrix}$$

are each equal to unity for every positive integer n.

SOLUTION BY C. F. GUMMER, Queen's University, Kingston, Ont.

Though this solution does not directly involve binomial coefficients, yet by finding the value of a_n it may be considered to dispose of the problem sufficiently.

The given conditions show that $a_0 = a_1 = 1$, and that the other a's may be found in succession uniquely from equations in which they appear with the coefficient unity. The a's being determinate there exists a sequence $x_0, x_1, \cdots$ such that

$$a_n = a_{n-1}x_0 + a_{n-2}x_1 + \cdots + a_0x_{n-1}, \qquad n = 1, \; , \cdots; \tag{1}$$

for the first n equations of (1) have a determinant equal to unity.

If we apply the substitutions (1) to the last row of

$$\begin{vmatrix} a_0 & a_1 & a_2 & \cdot & \cdot \\ a_1 & a_2 & \cdot & \cdot & \cdot \\ a_2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n & a_{n+1} & \cdot & \cdot & a_{2n} \end{vmatrix},$$

and simplify by means of the other rows, the last row becomes

$$0,\ a_0x_n,\ a_0x_{n+1} + a_1x_n,\ a_0x_{n+2} + a_1x_{n+1} + a_2x_n,\ \cdots.$$

With similar treatment, the preceding row becomes

$$0,\ a_0x_{n-1},\ a_0x_n + a_1x_{n-1},\ \cdots,$$

and so for all but the first row. On simplifying by columns we get, since $a_0 = 1$,

$$\begin{vmatrix} x_1 & x_2 & x_3 & \cdot & \cdot \\ x_2 & x_3 & \cdot & \cdot & \cdot \\ x_3 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & x_{2n-1} \end{vmatrix} = 1.$$

A like treatment of the other determinant gives

$$\begin{vmatrix} x_0 & x_1 & x_2 & \cdot & \cdot \\ x_1 & x_2 & \cdot & \cdot & \cdot \\ x_2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & x_{2n} \end{vmatrix} = 1.$$

Hence, $x_2, x_3, \cdots$ are defined in terms of x_0, x_1 in the same way as $a_2, a_3, \cdots$ in terms of a_0, a_1.

Also $a_0 = a_1 = 1$, $a_2 = 2$.

Hence, $x_0 = x_1 = 1$, $x_2 = 2$, by direct calculation.

Hence, $x_n = a_n$. Hence, (1) becomes

$$a_n = a_{n-1}a_0 + a_{n-2}a_1 + \cdots + a_0a_{n-1}, \qquad n = 1, 2, \cdots. \tag{2}$$

To calculate a_n, we infer from (2) that the coefficient of t^n in $u \equiv a_0 + a_1t + a_2t^2 + \cdots$ is equal to the coefficient of t^{n-1} in u^2, when $n = 1, 2, \cdots$.

Hence, $(u - 1)/t = u^2$.

Hence, $u = 1/2t - \sqrt{1 - 4t}/(2t)$, the minus sign being necessary to make u a series in positive powers of t.

Hence, the coefficient a_n of t^n in u equals $\dfrac{|\underline{2n}}{|\underline{n}\;|\underline{n+1}}$.